

# Low temperature decoherence by electron-electron interactions: Role of quantum fluctuations

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We derive a general expression for the conductivity of a disordered conductor with electron-electron interactions (treated within the standard model) and evaluate the weak localization correction  $\delta\sigma_{wl}$  employing no approximations beyond the accuracy of the definition of  $\delta\sigma_{wl}$ . Our analysis applies to all orders in the interaction and extends our previous calculation by explicitly taking into account quantum fluctuations around the classical paths for interacting electrons (pre-exponent). We specifically address the most interesting low temperature limit and demonstrate that such fluctuations can only be important in the perturbative regime of short times while they are practically irrelevant for the Cooperon dynamics at longer times. We fully confirm our conclusion about the existence of interaction-induced decoherence of electrons at zero temperature for the problem in question. We also demonstrate irrelevance of a perturbative calculation by Aleiner *et al.* (AAV) [J. Low Temp. Phys. **126**, 1377 (2002)] and refute AAV's critique of our earlier analysis.

## I. INTRODUCTION

The electron decoherence time in disordered conductors saturates to a finite value at low temperatures [1]. In Ref. [2] we offered an explanation of this effect attributing it to electron-electron interactions. This explanation was supported by a detailed nonperturbative calculation [3] and the results [2, 3] are in a good agreement with experimental findings [1]. Subsequently Aleiner *et al.* (AAG) developed an alternative – perturbative in the interaction – calculation [4] and claimed that the results of the latter (i) contradict to our results [2, 3] and (ii) yield zero electron dephasing rate at  $T \rightarrow 0$ . AAG also claimed (Sec. 6.1 and 6.2 of [4]) that they have found a “mistake” in our calculation [3].

All these claims [4] have been carefully analyzed and demonstrated to be in error [5]. We have argued that (a) the perturbative calculation [4] yields ambiguous results and, hence, is useless for the problem of electron dephasing at low temperatures, (b) even if one adopts the perturbative strategy [4] one recovers a *finite* electron dephasing time at  $T \rightarrow 0$ , (c) on a perturbative level our results [3] *do agree* with those of Ref. [4] and (d) the claim [4] about “missing diagrams” in our calculation is wrong. This discussion and further arguments on low temperature dephasing due to electron-electron interactions were reviewed in Ref. [6].

Recently Aleiner *et al.* (AAV) [7] have made another attempt to challenge our results and conclusions. One of the goals of the present paper is to analyze and refute these new criticisms. In particular, we will point out that the replacement of “the density matrix by its Wigner transform”, eq. (5) of [7], claimed by AAV as “the main source of the mistake” *was not performed at all* in our derivation of the effective action, cf. eq. (43) of Ref. [3]. We will also analyze and prove irrelevant another suggestion of AAV, that the term  $S_R$  in the effective action [3] should contain an imaginary part which – according to

AAV – would be responsible for the well-known cancellation of “coth” and “tanh” in the first order perturbation theory at  $T \rightarrow 0$ .

Since AAV's critique of our calculation is essentially restricted to these two claims [8, 9], the above observations are already sufficient to conclude this discussion. However, taking into account fundamental importance of the issue, we have performed an additional analysis aimed to construct a complete solution of the problem. Our main goal is to evaluate the weak localization (WL) correction to the conductivity in the presence of interactions making *no approximations* beyond the accuracy of the *definition* of this quantity. This solution is worked out below and our main result is presented in eqs. (40)-(42).

Our paper is organized as follows. After brief remarks on physics and experiment (Section 2) we demonstrate (in Section 3) that AAV's perturbative calculation [7] is unsuitable for the problem of quantum decoherence of electrons at low  $T$ . In Section 4 we briefly recollect our earlier results and, making use of general arguments, explain why our path integral analysis [3, 5, 6] is sufficient for the problem in question. The main results of this paper are presented in Section 5. We first derive a formally exact expression for the conductivity of an arbitrary disordered conductor in the presence of electron-electron interactions (Sec. 5A) and then use it to explicitly evaluate the WL correction to all orders in the interaction (Sec. 5B and 5C). This analysis confirms our previous results [3, 5] and extends them by fully accounting for quantum fluctuations around the classical paths for interacting electrons. We compare our results with those of other authors and present further discussion in Sec. 5D and 5E. In Section 6 we specifically address and refute AAV's critique [7] of our earlier calculation. A brief summary is presented in Section 7. Some technical details are relegated to Appendix.

## II. PHYSICS AND EXPERIMENT

In Ref. [7] AAV pointed out that our results and conclusions [2, 3, 5, 6] are “physically inconsistent” with the qualitative arguments [4] against quantum dephasing in the zero temperature limit. According to these authors quantum decoherence would be impossible in a subsystem of any interacting quantum system provided the latter is close to equilibrium at  $T \rightarrow 0$ . At this point we note that the arguments [4] contradict not only to our conclusions but also to numerous results for various other models – including exactly solvable ones – where quantum decoherence of one degree of freedom is obtained as a result of its interaction with others even at  $T \rightarrow 0$ , see, e.g., [10]. Hence, no general proof can be constructed which would rule out interaction-induced quantum decoherence at  $T \rightarrow 0$ . The conclusion about the presence or absence of zero temperature dephasing in any given model can only be obtained from a detailed calculation, not on the basis of “general arguments”.

AAV also claimed [7] that there exists an “overwhelming experimental evidence” against our statements. By means of a detailed comparison with the experimental data we have demonstrated that our predictions are in a good quantitative agreement with numerous experiments (see, e.g., Refs. [2, 3, 6, 11]) *including* those (see Section 5 of Ref. [6]) where for some samples no saturation of  $\tau_\varphi$  was claimed down to  $T \sim 50$  mK. Therefore, a bulk piece of existing experimental results – including very recent ones [12, 13] – clearly supports the conclusion about the presence of interaction-induced dephasing at low  $T$  rather than argues against it.

## III. INSUFFICIENCY OF AAV’S PERTURBATIVE APPROACH

In Refs. [5, 6] we have already explained in details why perturbative in the interaction techniques – especially if combined with the Golden rule approximation – are insufficient for the problem in question. However, since AAV still keep using such techniques, we will briefly repeat our arguments adopting them to the calculation [7].

In Ref. [7] AAV have related the dephasing time  $\tau_\varphi$  for an electron in a disordered conductor to the self-energy for the Cooperon. Expressing the Cooperon in the form

$$\mathcal{C}(\omega, Q, \epsilon) = \frac{1}{-i\omega + DQ^2 + \Sigma(\omega, Q, \epsilon)}, \quad (1)$$

they have defined the dephasing time at  $T = 0$  as follows

$$\tau_\varphi^{-1} = \Sigma(\omega = 0, Q = 0, \epsilon), \quad (2)$$

see the unnumbered equation after eq. (2) of [7]. This definition is ambiguous since the self-energy is a function of  $\epsilon$  whereas  $\tau_\varphi$  in our problem is a function of temperature  $T$  but not of  $\epsilon$ . The ambiguity disappears only at

$T \rightarrow 0$ , since according to Eq. (1) of Ref. [7] in this limit one should set  $\epsilon \rightarrow 0$ . Another drawback of the definition (2) is that the self-energy is evaluated only in the limit  $Q = 0$ . However, the dependence of  $\Sigma$  on the momentum  $Q$  in eq. (1) cannot be *a priori* neglected.

In addition to the above approximations, AAV suggested to replace the exact self-energy in Eq. (2) by the result of the first order perturbation theory in the interaction  $\Sigma^{(1)}(\omega, Q, \epsilon)$ :

$$\frac{1}{\tau_\varphi^{\text{AAV}}} = \Sigma^{(1)}(\omega = 0, Q = 0, \epsilon = 0). \quad (3)$$

This suggestion constitutes their *major mistake*. Quite obviously, it is not possible to recover the unknown function (in our case  $\Sigma$ ) if one only evaluates the first order term ( $\Sigma^{(1)}$ ) of its Taylor expansion.

In order to illustrate this point it suffices to consider the following simple example. Assume, for instance, that the “Cooperon”  $\tilde{C}(t)$  depends weakly on the coordinates and neglect this dependence. We also assume that in the presence of interactions this “Cooperon” decays in time as

$$\tilde{C}(t) = \theta(t)(1 + \alpha t)e^{-(\alpha + \beta T)t}, \quad (4)$$

where  $\alpha$  and  $\beta$  are proportional to the interaction strength, i.e (4) reduces to  $\theta(t)$  in the absence of interactions. After the Fourier transformation of (4) one readily finds

$$\begin{aligned} \tilde{C}(\omega) &= \frac{1}{-i\omega + \tilde{\Sigma}(\omega)}, \\ \tilde{\Sigma}(\omega) &= \frac{(\alpha + \beta T)^2 - i\omega\beta T}{2\alpha + \beta T - i\omega}. \end{aligned} \quad (5)$$

Combining (5) with the definition (2) we obtain

$$\frac{1}{\tau_\varphi} = \frac{\alpha + \beta T}{1 + \frac{\alpha}{\alpha + \beta T}}, \quad (6)$$

i.e. at  $T \rightarrow 0$  one arrives at a non-zero dephasing rate

$$\tau_\varphi^{-1} = \alpha/2. \quad (7)$$

Let us now evaluate  $\tau_\varphi$  for the same example following the approach of AAV. For this purpose we expand the exact expression for  $\tilde{\Sigma}(\omega)$  (5) in powers of interaction and get

$$\tilde{\Sigma}(\omega) = \beta T + \frac{\alpha^2}{-i\omega} + \dots \quad (8)$$

Keeping only the first order contribution to the self-energy, from AAV’s eq. (3) we find

$$\frac{1}{\tau_\varphi^{\text{AAV}}} = \beta T. \quad (9)$$

This result differs drastically from the exact one (6) at sufficiently low temperatures. In particular, at  $T \rightarrow 0$  the

dephasing rate (9) vanishes, while the exact expression approaches a non-zero (linear in the interaction) value (7). Furthermore, all higher order terms in the expansion (8) do not vanish at  $T \rightarrow 0$  and, moreover, diverge at small  $\omega$ .

The reason for the failure of AAV's perturbative approach is, of course, obvious from eq. (5): An expansion of this expression in powers of  $\alpha$  and  $\beta$  is only justified for  $\omega \gg \alpha + \beta T$ , i.e. in the limit of high frequencies or short times. We, in contrast, are interested in the opposite limit of small frequencies or long times. Thus, AAV simply missed the low temperature contribution to the dephasing rate by inadequately extending their perturbative expansion of  $\Sigma(\omega, Q, \epsilon)$  to low frequencies, whereas it can be applied at high frequencies only.

Our simple example, eq. (5), also illustrates irrelevance of the claim [7] that our approach “is equivalent to calculating only a single contribution  $\Sigma^{(b)}$  to the self-energy and using the conventional Dyson equation”. AAV arrived at this conclusion simply by observing the combination  $\Sigma^{(b)} - \Sigma^{(c-f)} \sim \beta T$  in their first order perturbative result (cf. eqs. (3,4) in [7]) and comparing it with our result, which contains not only  $\beta T$  but also the  $T$ -independent contribution. The only – superficial – reason for Aleiner *et al.* to perform this comparison and to qualify our results as “purely perturbative” is that our expression for the dephasing rate “is proportional to the first power of the fluctuation propagator” [4]. We hope it should be sufficiently clear from eq. (5) that *linear* in the interaction expression (7) is *non-perturbative* and, hence, it cannot be obtained by a simple expansion (8) of the self-energy  $\tilde{\Sigma}(\omega)$  in the interaction.

In order to avoid misunderstandings [14] let us emphasize that the above example (4) is *not* meant to be an explicit solution for the problem with electron-electron interactions. This solution will be worked out below in Sec. 5. Eq. (4) is just an illustration of one of the drawbacks of the perturbative approach [7] to the problem of quantum dephasing. In the problem with disorder and electron-electron interactions the situation turns out to be by far more complicated. For instance, already the first order result diverges both with time and at large frequencies in 1d and 2d systems, see, e.g., eq. (70) of Ref. [5].

#### IV. PATH INTEGRAL ANALYSIS: EXPONENT

An important advantage of our path integral approach is the possibility to describe the long-time behavior of the Cooperon with exponential accuracy, which is sufficient for the problem in question. This approach is free from ambiguities inherent to the perturbation theory [4, 7].

We define the kernel of the evolution on the Keldysh contour in terms of the path integral [3, 5, 6]

$$\int \mathcal{D}p_1 \int \mathcal{D}x_1 \int \mathcal{D}p_2 \int \mathcal{D}x_2$$

$$\times e^{\frac{1}{\hbar}(iS_0[p_1, x_1] - iS_0[p_2, x_2] - iS_R[p_1, x_1, p_2, x_2] - S_I[x_1, x_2])} \quad (10)$$

Here  $S_0[p, x] = \int_0^t dt' [p\dot{x} - p^2/2m - U_{\text{imp}}(x)]$  is the action of a noninteracting electron in a disordered potential of impurities and the terms  $iS_R$  and  $S_I$  describe the effect of the bath (formed by all the electrons) on the motion of a single electron. This form of the effective action for an interacting particle is standard in the Feynman-Vernon theory of influence functionals [15]. The path integral (10) is evaluated within a semiclassical approximation controlled by the parameter  $k_F l \gg 1$ . For a  $d$ -dimensional conductor we find

$$\mathcal{C}(t, x=0) = \theta(t) \frac{A_d(t)}{(4\pi Dt)^{d/2}} e^{-f_d(t)}, \quad (11)$$

where  $D$  is the diffusion coefficient,  $A_d(t)$  is pre-exponent ( $A_d(t) \equiv 1$  without interactions), and the function  $f_d(t)$  is obtained by evaluating the influence functional on pairs of classical time-reversed paths. The result can be written in the form [5]

$$f_d(t) = \alpha t + \delta f_d(T, t), \quad (12)$$

where the function  $\delta f_d(T, t)$  is  $\propto Tt^{3/2}$  for  $d = 1$  and  $\propto Tt \ln Tt$  for  $d = 2$  in the limit  $Tt \gg 1$ , while it is  $\propto \sqrt{t} \ln t$  for  $d = 1$  and  $\propto \ln t$  in the opposite limit  $Tt \ll 1$ . The linear dependence  $\delta f_d(T, t) \propto Tt$  strictly applies for  $d = 3$ .

It is also instructive to explicitly indicate the dependence of our results on the Planck's constant  $\hbar$ . At  $T \rightarrow 0$  we have

$$\ln[\mathcal{C}_d(t)/\mathcal{C}_d^{(0)}(t)] = -\frac{S^{(\text{cl})}(t)}{\hbar} + \ln[A_d(\hbar, t)], \quad (13)$$

where  $S^{(\text{cl})} \simeq at$  is the classical ( $\hbar$ -independent) action on time-reversed saddle point paths ( $a$  is  $\hbar$ -independent and  $\alpha \equiv a/\hbar$ ) and  $\hbar \ln A_d$  represents the quantum correction to the classical action. This quantum correction can only be important if  $S^{(\text{cl})}$  is small as compared to  $\hbar$ , i.e. at times  $t \ll \hbar/a$ . The perturbative approach [4, 7] applies *only* in this limit. On the other hand, for nonzero  $t$  the quantum correction  $\hbar \ln A_d$  should vanish at  $\hbar \rightarrow 0$ . Already because of this reason it cannot cancel the classical part of the action  $S^{(\text{cl})} \gtrsim \hbar$ . It is therefore sufficient to evaluate the  $\hbar$ -independent part of  $-iS_R - S_I$  on pairs of classical time-reversed paths and obtain the dephasing time from the condition  $S^{(\text{cl})}(\tau_\varphi) \sim \hbar$ . Furthermore, it turns out that  $S_R$  vanishes for such paths [3] implying that  $S_R$  can only contribute to the *pre-exponent* but not to  $S^{(\text{cl})}$ . Hence,  $S_R$  is irrelevant for  $\tau_\varphi$  and the idea that quantum fluctuations around the time-reversed classical paths generate the  $S_R$ -dependent contribution to the classical action can be rejected on general grounds without any calculation [16].

This is the logic behind our saddle point analysis [3, 5] which AAV [7] attempted to challenge. The only way to support the AAV's arguments is to prove that at  $T \rightarrow 0$  the term  $S_R$  provides a contribution to  $A_d$  proportional

to  $\exp(at/\hbar)$  which grows *exponentially* with time and *diverges* in the classical limit  $\hbar \rightarrow 0$  [in Ref. [7] such a contribution was claimed to be provided by the terms  $\Sigma^{(c-f)}$ ]. Below we will explicitly evaluate not only  $S^{(cl)}$  but also the quantum correction to the effective action in all orders in the interaction. We will demonstrate that – in contrast to the AAV’s claims – the  $S_R$ -dependent pre-exponent cannot grow at sufficiently long times and, hence, in no way can compensate an exponentially decaying contribution from the  $S_I$ -terms.

## V. DECOHERENCE BY INTERACTIONS

### A. Exact results

In the beginning we will closely follow the analysis of Ref. [3] where the interested reader can find further details. We start from the general quantum mechanical expression for the linear conductivity  $\sigma$  which can be written in the form

$$\sigma = \frac{e}{3i\hbar} \int_{-\infty}^t dt' \left\langle \text{tr} \left( \hat{j}(x) \hat{U}_1(t, t') [\hat{x}, \hat{\rho}_V(t')] \hat{U}_2(t', t) \right) \right\rangle_V \quad (14)$$

Here the current density operator is defined via

$$\langle x_1 | \hat{j}(x) | x_2 \rangle = \frac{\hbar e}{im} [\nabla_{x_1} \delta_{x_1, x} \delta_{x_2, x} - \delta_{x_1, x} \nabla_{x_2} \delta_{x_2, x}] \quad (15)$$

and  $\langle \dots \rangle_V$  implies averaging over the fluctuating quantum fields  $V^+$  and  $V^-$  which mediate Coulomb interaction between electrons. In eq. (14) and below we implicitly assume averaging over the coordinate  $x$  where the current density is evaluated.

The evolution operators  $\hat{U}_{1,2}$  in (14) are

$$\hat{U}_{1,2}(t, t') = \mathbf{T} \exp \left[ -\frac{i}{\hbar} \int_{t'}^t d\tau \hat{H}_{1,2}(\tau) \right], \quad (16)$$

where

$$\begin{aligned} \hat{H}_1(t) &= \hat{H}_0 - \mu - e\hat{V}^+(t) - \frac{1}{2}[1 - 2\hat{\rho}_V(t)]e\hat{V}^-(t), \\ \hat{H}_2(t) &= \hat{H}_0 - \mu - e\hat{V}^+(t) + \frac{1}{2}e\hat{V}^-(t)[1 - 2\hat{\rho}_V(t)], \end{aligned} \quad (17)$$

and  $\hat{H}_0 = \hat{p}^2/2m + U_{\text{imp}}(\hat{x})$  is the Hamiltonian of a non-interacting electron. The density matrix  $\rho_V(t')$  obeys the non-linear equation

$$\begin{aligned} i\hbar \frac{\partial \hat{\rho}_V}{\partial t} &= [\hat{H}_0 - e\hat{V}^+, \hat{\rho}_V] - \frac{1}{2}(1 - \hat{\rho}_V)e\hat{V}^- \hat{\rho}_V \\ &\quad - \frac{1}{2}\hat{\rho}_V e\hat{V}^-(1 - \hat{\rho}_V). \end{aligned} \quad (18)$$

Averaging of  $\rho_V(t)$  over  $V^\pm$  yields the exact single electron density matrix in the presence of disorder and Coulomb interactions. The next steps in Refs. [3, 6]

were to express the kernels of the operators  $\hat{U}_{1,2}$  in terms of the path integrals and average their product over the fluctuating fields  $V^\pm$ . After that one arrives at the path integral (10).

At this point we depart from the analysis of Refs. [3, 6]. We will postpone using the path integrals and continue exact manipulations with the operators. We first note that the solution of eq. (18) with the initial condition  $\hat{\rho}_V(0) = \hat{\rho}_0 \equiv [1 + e^{(\hat{H}_0 - \mu)/T}]^{-1}$  can be expressed in the following exact form

$$\hat{\rho}_V(t) = \left[ 1 + \hat{u}_2(t, 0) e^{(\hat{H}_0 - \mu)/T} \hat{u}_1(0, t) \right]^{-1}, \quad (19)$$

where we have defined

$$\hat{u}_{1,2}(t, t') = \mathbf{T} \exp \left[ -\frac{i}{\hbar} \int_{t'}^t d\tau \hat{\tilde{H}}_{1,2}(\tau) \right] \quad (20)$$

and

$$\hat{\tilde{H}}_{1,2} = \hat{H}_0(t') - \mu - e\hat{V}^+(t') \mp e\hat{V}^-(t')/2. \quad (21)$$

We then observe that the operators  $\hat{U}_{1,2}$  satisfy the Schrödinger equation

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \hat{U}_1(t, t') &= \hat{H}_1(t) \hat{U}_1(t, t'), \quad \hat{U}_1(t', t') = \hat{1}, \\ i\hbar \frac{\partial}{\partial t} \hat{U}_2(t', t) &= \hat{U}_2(t', t) \hat{H}_2(t), \quad \hat{U}_2(t', t') = \hat{1}. \end{aligned} \quad (22)$$

The solutions of (22) can be found exactly. They are

$$\begin{aligned} \hat{U}_1(t, t') &= [1 - \hat{\rho}_V(t)] u_1(t, t') [1 - \hat{\rho}_V(t')]^{-1}, \\ \hat{U}_2(t', t) &= [1 - \hat{\rho}_V(t')]^{-1} u_2(t', t) [1 - \hat{\rho}_V(t)]. \end{aligned} \quad (23)$$

Combining these expressions with eq. (19) one can rewrite the operators (23) in the following identical form

$$\begin{aligned} \hat{U}_1(t, t') &= [(1 - \hat{\rho}_V(t')) \hat{u}_1(t', t) + \hat{\rho}_V(t') \hat{u}_2(t', t)]^{-1}, \\ \hat{U}_2(t', t) &= [\hat{u}_2(t, t') (1 - \hat{\rho}_V(t')) + \hat{u}_1(t, t') \hat{\rho}_V(t')]^{-1}. \end{aligned} \quad (24)$$

Eqs. (24) is our key technical result. Together with eqs. (14,15) this result provides a formally exact expression for the linear conductivity of an arbitrary disordered conductor in the presence of electron-electron interactions. *All* the diagrams of the perturbation theory in *all* orders in the interaction are fully contained in the above expressions and can be recovered by expanding (24) in  $V^\pm$  with subsequent averaging over these fields. For instance, expanding (24) to the second order in  $V^\pm$ , after averaging one arrives at contributions of the type  $\langle V^+ V^+ \rangle$  and  $\langle V^+ (1 - 2\hat{\rho}) V^- \rangle$  which yield respectively “coth” and “tanh” terms in the perturbation theory [4, 5].

Thus, making no approximations we have demonstrated that the time evolution of the single electron density matrix in the presence of interactions is determined by the operators  $\hat{u}_{1,2}(t, t')$  (20) which do not contain the density matrix  $\hat{\rho}_V$  at all. We also note that the operators  $\hat{U}_{1,2}$  (24) depend on the density matrix  $\hat{\rho}_V$  taken at the initial time  $t'$  only but not at later time moments. Below we will make use of these features and evaluate the WL correction to conductivity to all orders in the interaction.

## B. Quasiclassics

Let us first prepare the main building blocks of our calculation of the WL correction. This calculation will then be completed in Sec. 5C.

To begin with, we notice that in eqs. (24) relative contributions of the evolution operators  $\hat{u}_{1,2}(t, t')$  are “weighted” by the factors  $1 - \hat{\rho}_V(t')$  and  $\hat{\rho}_V(t')$ . In the low temperature limit almost at any electron energy one of these factors dominates over the other [19]. Hence, one of the two operators  $\hat{u}_1$  or  $\hat{u}_2$  in (24) can be neglected except if the eigenvalues of  $\hat{\rho}_V(t')$  are close to  $1/2$ . Consider, e.g., small eigenvalues of this operator. This situation describes electrons with energies above the Fermi level. In this case terms containing  $\hat{\rho}_V(t')$  in (24) can be neglected and we arrive at the following contribution to  $\langle x_1 | \hat{U}_1(t, t') [\hat{x}, \hat{\rho}_V(t')] \hat{U}_2(t', t) | x_2 \rangle$ :

$$\int \int dz_1 dz_2 \langle x_1 | \hat{u}_1(t, t') | z_1 \rangle \langle z_2 | \hat{u}_2(t', t) | x_2 \rangle \times \langle z_1 | [\hat{x}, \hat{\rho}_V(t')] | z_2 \rangle. \quad (25)$$

Let us express the matrix elements of the operators  $\hat{u}_{1,2}$  via the path integrals

$$\langle y_{1,2} | \hat{u}_{1,2}(t, t') | z_{1,2} \rangle = \int_{y_{1,2}}^{z_{1,2}} \mathcal{D}x(\tau) e^{\frac{i}{\hbar} S_{1,2}[x(\tau)]}, \quad (26)$$

where  $S_{1,2}$  are the exact actions pertaining to the Hamiltonians (21)

$$S_{1,2} = S_0 + e \int_{t'}^t d\tau [V^+(\tau, x(\tau)) \pm V^-(\tau, x(\tau)) / 2], \quad (27)$$

$$S_0 = \int_{t'}^t d\tau \left( \frac{m\dot{x}^2(\tau)}{2} - U_{\text{imp}}(x(\tau)) \right). \quad (28)$$

We emphasize again that eqs. (26-28) are exact and they do not contain the electron density matrix  $\hat{\rho}_V$  at all. We should now (a) evaluate the path integrals (26) and then (b) average the combination (25) over the fluctuating fields  $V^\pm$ .

*Evaluation of the matrix elements.* Let us make use of the fact that the WL correction to conductivity is defined within the accuracy  $k_F l \gg 1$ . This inequality is usually well satisfied in disordered metallic conductors. Hence, we can evaluate the matrix elements (26) quasiclassically. Since the actions (27) do not depend on  $\rho_V$  one can conveniently employ a regular expansion of  $S_{1,2}$  in powers of  $\hbar$ . The path integrals are then easily evaluated and we arrive at the well-known van Vleck formula

$$\langle x_{1,2} | \hat{u}_{1,2}(t, t') | z_{1,2} \rangle = \sum_n \sqrt{\left( \frac{i}{2\pi\hbar} \right)^3 \det \frac{\partial^2 S_{1,2}^{(n)}}{\partial x_{1,2} \partial z_{1,2}}} e^{\frac{i}{\hbar} S_{1,2}^{(n)}}, \quad (29)$$

where  $S_{1(2)}^{(n)} \equiv S_{1(2)}^{(n)}(t, t', x_{1(2)}, z_{1(2)}) = S_{1(2)}[\tilde{x}_{1n(2n)}]$  and

$\tilde{x}_{1,2}$  are the exact least action paths obeying the equations

$$\delta S_{1(2)}[\tilde{x}_{1n(2n)}, V^\pm] / \delta \tilde{x}_{1n(2n)} = 0 \quad (30)$$

with the boundary conditions  $\tilde{x}_{1n(2n)}(t') = x_{1(2)}$  and  $\tilde{x}_{1n(2n)}(t) = z_{1(2)}$ . In general there exist several or even many different classical paths satisfying the above conditions. Here and below the index  $n$  labels all such paths.

Clearly, eq. (29) accounts not only for the saddle point trajectories (exponent) but also for quantum fluctuations around  $\tilde{x}_{1n,2n}(\tau)$  (pre-exponent) for arbitrary  $V^\pm$ . It is also completely obvious that there is no way how the pre-exponent can cancel the exponent for any configuration of the fields  $V^\pm$ . Hence, such cancellation is impossible also after averaging over these fields no matter what the details of this averaging procedure are.

*Averaging over the fluctuating fields.* In general averaging of the combination (25) over  $V^\pm$  involves path integrals (A1) over these fields at all times from zero to  $t$ . This is because  $\hat{\rho}_V(t')$  is nonlocal in time: According to eq. (19) it depends on times between zero and  $t'$ . However, with the accuracy  $k_F l \gg 1$  one can perform averaging in (25) at times smaller and larger than  $t'$  separately. This splitting is achieved by expressing  $\rho_V(t')$  (19) via the path integrals, making use of eqs. (26-28) and averaging the whole combination (25) over  $V^\pm$  at all times between 0 and  $t$ . One arrives at the effective actions containing nonlocal in time contributions  $S_{R,I}(t_1, t_2)$  which vanish for all the trajectories relevant in the quasiclassical limit  $k_F l \gg 1$  provided  $t_1 > t'$ ,  $t_2 < t'$  and vice versa. As a result we obtain

$$\int \int dz_1 dz_2 J_{12}(t, t'; x_{1,2}, z_{1,2}) (z_1 - z_2) \rho(z_1, z_2), \quad (31)$$

$$J_{12} = \langle \langle x_1 | \hat{u}_1(t, t') | z_1 \rangle \langle z_2 | \hat{u}_2(t', t) | x_2 \rangle \rangle_V. \quad (32)$$

Here we used  $\langle z_1 | [\hat{x}, \hat{\rho}] | z_2 \rangle \equiv (z_1 - z_2) \rho(z_1, z_2)$ , where  $\hat{\rho} = \langle \hat{\rho}_V(t') \rangle_V$  is the exact equilibrium electron density matrix in the presence of interactions. We can also add that, as it was explained in Sec. 4 of Ref. [6], with the same accuracy  $k_F l \gg 1$  one can replace

$$\hat{\rho}_V(t') \longrightarrow \hat{\rho}_0 = [1 + e^{(\hat{H}_0 - \mu)/T}]^{-1} \quad (33)$$

already before averaging over  $V^\pm$ . After that the factorization (31) (with  $\rho \rightarrow \rho_0$ ) is, of course, an exact procedure.

What remains is to average the product of the two matrix elements in (32). This averaging is carried out in a standard manner. If the fields  $V^\pm$  vary at scales exceeding the elastic mean free path, one can neglect the dependence of both the classical paths  $\tilde{x}_{1,2}$  and the pre-exponent in eq. (29) on the fields  $V^\pm$ . This approximation is sufficient for evaluation of the WL correction to the conductivity.

Then averaging with the action (A2) can be performed exactly. We first integrate over  $V^+$ . As both actions

(A2) and (27) are linear in  $V^+$ , this integration yields the  $\delta$ -function

$$\delta(V^-(\tau, x) - V_0(\tau, x, \tilde{x}_{1n}(s), \tilde{x}_{2m}(s))),$$

where

$$V_0(\tau, x, \tilde{x}_{1n}(s), \tilde{x}_{2m}(s)) = -e \int_{t'}^t ds [R(s - \tau, \tilde{x}_{1n}(s) - x) - R(s - \tau, \tilde{x}_{2m}(s) - x)] \quad (34)$$

and the function  $R(t, x)$  is defined in (A4). Due to this  $\delta$ -function the subsequent integration over  $V^-$  also becomes trivial and we obtain  $J_{12} = \sum_{n,m} J_{12}^{nm}$ , where

$$J_{12}^{nm} = \frac{1}{(2\pi\hbar)^3} \sqrt{\det \frac{\partial^2 S_0^{(n)}}{\partial x_1 \partial z_1} \det \frac{\partial^2 S_0^{(m)}}{\partial x_2 \partial z_2}} \times e^{\frac{1}{\hbar} [iS_0[\tilde{x}_{1n}] - iS_0[\tilde{x}_{2m}] - i\tilde{S}_R[\tilde{x}_{1n}, \tilde{x}_{2m}] - S_I[\tilde{x}_{1n}, \tilde{x}_{2m}]]}. \quad (35)$$

In eq. (35) the term  $S_I$  is identical to one derived in Ref. [3] (see eq. (55) of that paper) while the action  $\tilde{S}_R$  is obtained from eq. (54) of [3] by formally setting the function  $n(\mathbf{p}, \mathbf{r})$  equal to zero in that formula. The action  $\tilde{S}_R$  is purely real for any pair of paths  $\tilde{x}_{1n}$  and  $\tilde{x}_{2m}$ . Thus, together with the terms  $iS_0$  it can only provide oscillations of the kernel  $J$  and in no way can compensate its decay  $J_{12}^{nm} \propto \exp(-S_I/\hbar)$ . As it was already discussed in Ref. [3] and elsewhere, the action  $S_I$  is real and positive for *any* pair of trajectories (except for identical ones in which case  $S_I = 0$ ). The length of electron trajectories in a metal always grows with time since electrons move with a constant velocity  $\sim v_F$ . Hence, for any pair of time-reversed paths  $\tilde{x}_{1n}(s) = \tilde{x}_{2m}(t + t' - s)$  the action  $S_I$  grows with time as well. This in turn implies that for such paths the kernel  $J_{12}^{nm}$  decays with time and vanishes in the long time limit  $t - t' \rightarrow \infty$  at any temperature including, of course,  $T = 0$ .

In the above analysis we neglected the evolution operator  $\hat{u}_2$  ( $\hat{u}_1$ ) in the exact expression for  $\hat{U}_1$  ( $\hat{U}_2$ ) (24). This is correct at low  $T$  and for the electron energies above the Fermi level. Below the Fermi energy, on the contrary, one can drop terms containing  $1 - \hat{\rho}_V(t')$  because in this case the eigenvalues of  $\hat{\rho}_V(t')$  are close to one. Then the whole analysis is repeated, one should only interchange the operators  $\hat{u}_1$  and  $\hat{u}_2$ . In this way one again arrives at eq. (35) (with  $\tilde{S}_R \rightarrow -\tilde{S}_R$ ) which again decays as  $\exp(-S_I/\hbar)$ . The remaining options are to neglect either  $\hat{u}_1$  or  $\hat{u}_2$  in both expressions for  $\hat{U}_{1,2}$  (24). One again finds the contributions  $\propto \exp(-S_I/\hbar)$ . Since in all these cases  $S_I$  remains the same, one concludes that if the operators  $\hat{u}_1$  and  $\hat{u}_2$  yield comparable contributions (in which case the exact form of (24) should be used) they will also decay as  $\exp(-S_I/\hbar)$  on any pair of time-reversed paths. Below we present an explicit calculation of the WL correction which fully confirms this conclusion.

### C. Pre-exponent and weak localization correction

We are now prepared to evaluate the conductivity. As before, we assume that  $k_F l \gg 1$  and that the fields  $V^\pm$  vary in space at scales exceeding the elastic mean free path  $l$ . In this case quasiclassical electron trajectories are not disturbed by interactions, and the contributions of the fluctuating fields  $V^\pm$  add up independently. Therefore, we can approximately split the operators

$$\hat{u}_{1,2}(t, t') \simeq \hat{u}_0(t, t') \hat{s}(t, t', V^+) \hat{s}(t, t', \pm V^-/2), \quad (36)$$

where  $\hat{u}_0(t, t')$  is the evolution operator pertaining to the non-interacting Hamiltonian and

$$\hat{s}(t, t', V) = \hat{u}_0(t', t) \mathbf{T} \exp \left[ -\frac{i}{\hbar} \int_{t'}^t d\tau (\hat{H}_0 - e\hat{V}(\tau)) \right] \quad (37)$$

Within the same accuracy we can replace

$$\begin{aligned} \hat{u}_0(t, t') \hat{s}(t, t', V^+) &\simeq \hat{u}(t, t', V^+) \\ &= \mathbf{T} \exp \left( -\frac{i}{\hbar} \int_{t'}^t d\tau [\hat{H}_0 - e\hat{V}^+(\tau)] \right). \end{aligned} \quad (38)$$

Combining (36)-(38) with (24) we obtain

$$\begin{aligned} \hat{U}_1(t, t') &\simeq \hat{u}(t, t', V^+) \{ (1 - \hat{\rho}_V(t')) \hat{s}^{-1}(t, t', V^-/2) \\ &\quad + \hat{\rho}_V(t') \hat{s}^{-1}(t, t', -V^-/2) \}^{-1}, \\ \hat{U}_2(t', t) &\simeq \{ \hat{s}(t, t', -V^-/2) (1 - \hat{\rho}_V(t')) \\ &\quad + \hat{s}(t, t', V^-/2) \hat{\rho}_V(t') \}^{-1} \hat{u}(t', t, V^+). \end{aligned} \quad (39)$$

Substituting (39) into eq. (14), evaluating the matrix elements of the operator  $u(t, t', V^+)$  by means of the van Vleck formula (29) and integrating over the fluctuating fields  $V^\pm$  exactly as in Sec. 5B [20], we find

$$\begin{aligned} \sigma &= \frac{e^2}{3m} \sum_{n,m} \int_{-\infty}^t dt' \int dy_1 dy_2 \int dz_1 dz_2 \\ &\quad \times (\nabla_{x_1} - \nabla_{x_2})|_{x_1=x_2} J^{nm}(t, t'; x_1, x_2; y_1, y_2) \\ &\quad \times A_1^{nm}(t, t', y_1, z_1, x_1, x_2) (z_1 - z_2) \rho_0(z_1, z_2) \\ &\quad \times A_2^{nm}(t, t', z_2, y_2, x_1, x_2), \end{aligned} \quad (40)$$

where

$$\begin{aligned} J^{nm} &= \frac{1}{(2\pi\hbar)^3} \sqrt{\det \frac{\partial^2 S_0^{(n)}}{\partial x_1 \partial y_1} \det \frac{\partial^2 S_0^{(m)}}{\partial x_2 \partial y_2}} \\ &\quad \times \exp \left\{ \frac{i}{\hbar} S_0[\tilde{x}_{1n}] - \frac{i}{\hbar} S_0[\tilde{x}_{2m}] - \frac{1}{\hbar} S_I[\tilde{x}_{1n}, \tilde{x}_{2m}] \right\} \end{aligned} \quad (41)$$

and

$$\begin{aligned} A_1^{nm} &= \langle y_1 | \{ (1 - \hat{\rho}_0) \hat{s}^{-1}(t, t', V^-/2) \\ &\quad + \hat{\rho}_0 \hat{s}^{-1}(t, t', -V^-/2) \}^{-1} | z_1 \rangle |_{V^- = V_0(\tau, x, \tilde{x}_{1n}, \tilde{x}_{2m})}, \\ A_2^{nm} &= \langle z_2 | \{ \hat{s}(t, t', -V^-/2) (1 - \hat{\rho}_0) \\ &\quad + \hat{s}(t, t', V^-/2) \hat{\rho}_0 \}^{-1} | y_2 \rangle |_{V^- = V_0(\tau, x, \tilde{x}_{1n}, \tilde{x}_{2m})} \end{aligned} \quad (42)$$

As before, the paths  $\tilde{x}_{1n}$  and  $\tilde{x}_{2m}$  satisfy the Newton equation (30) (with  $V^\pm = 0$ ) and the boundary conditions  $\tilde{x}_{1n(2m)}(t') = y_{1(2)}$  and  $\tilde{x}_{1n(2m)}(t) = x_{1(2)}$ .

Eqs. (40)-(42) represent the central result of this paper. They determine the linear conductivity of an arbitrary disordered conductor to all orders in the electron-electron interaction. The above equations are based on the exact results (24) and are valid in the quasiclassical limit  $k_F l \gg 1$ . We would like to emphasize that no quasiclassical approximation for the electron density matrix was employed during our derivation and no averaging over impurities was performed at all.

Let us briefly analyze eqs. (40)-(42). According to the standard arguments two types of classical paths  $\tilde{x}_{1n}$  and  $\tilde{x}_{2m}$ , identical and time-reversed ones, play an important role in the quasiclassical limit  $k_F l \gg 1$ . For a pair of identical paths  $\tilde{x}_{1n}(s) = \tilde{x}_{2n}(s)$  the two actions  $S_0$  in the exponent (41) cancel each other, the term  $S_I$  vanishes identically and the matrix elements  $A_{1,2}$  reduce to  $\delta$ -functions  $A_{1,2}^{nm} = \delta(y_{1,2} - z_{1,2})$  because  $V_0 \equiv 0$  in this case. In this way we recover the well known property that the diffuson does not decay in time even in the presence of interactions.

Here we are interested in the quantum correction to conductivity arising from the time-reversed paths  $\tilde{x}_{1n}(s) = \tilde{x}_{2m}(t + t' - s)$ . For any pair of such paths the actions  $S_0$  cancel again but the interaction term  $S_I$  is now positive, it grows with time and yields (exponential) decay of the quantity  $J^{nm}$  (41) in the long time limit. The matrix elements  $A_{1,2}^{nm}$  also depend on the interaction and on time in this case. It is obvious, however, that  $A_{1,2}^{nm}$  *cannot grow* at long times because the function  $V_0$  (34) is purely real and, hence,  $\hat{s}(t, t', \pm V_0/2)$  are the unitary operators. The matrix elements of such operators can only oscillate provided the function  $V_0$  changes in time. Hence, no compensation of decaying  $J^{nm} \propto \exp(-S_I(t - t')/\hbar)$  can be expected for sufficiently large  $t - t'$ , and the whole expression under the integral over  $t'$  in eq. (40) decays exponentially together with  $J^{nm}$  for any pair of time-reversed paths. Obviously, the matrix elements (42) also cannot grow if one formally takes the limit  $\hbar \rightarrow 0$ , while  $J^{nm}$  vanishes in this limit. All that implies that – in full agreement with our general arguments (Sec. 4) – one can indeed obtain the dephasing time from the condition  $S_I(\tau_\varphi) \sim \hbar$  which does not depend on the density matrix  $\hat{\rho}_V$  and identically reproduces our earlier results [2, 3, 5, 6].

Although in principle one can proceed further and under certain approximations evaluate the matrix elements (42) for pairs of time-reversed paths, we will not do it here. The reason for that is obvious from the above discussion: Particular values of  $A_{1,2}^{nm}$  are irrelevant for dephasing. It suffices to observe that these matrix elements do not grow at long times.

## D. Relation to other results

It is useful to compare eqs. (40)-(42) with the results of some earlier calculations of the WL correction. Altshuler, Aronov and Khmelnitskii (AAK) [21] considered the electron dephasing by a fluctuating external field. Their results are easily recovered from our calculation if one sets  $V^- \equiv 0$ . Then one again arrives at eqs. (40)-(42) with  $A_{1,2}^{nm} \equiv \delta(y_{1,2} - z_{1,2})$ . AAK furthermore applied their results to the problem of interacting electrons identifying an external field with one produced by fluctuating electrons (the field  $V^+$  in our analysis). In order to account for Pauli blocking AAK suggested a phenomenological procedure which amounts to keeping only the classical part of this field and to cutting out its quantum modes with frequencies  $\omega > T$  (i.e. all modes at  $T \rightarrow 0$ ). This last step has no analogy in our calculation.

An attempt to justify the procedure [21] was recently undertaken by AAG [4] within the framework of the first order perturbation theory in the interaction. Our general expressions, if expanded to the first order, yield eq. (A6) which can be written in the form

$$\sigma = \sigma^{(0)} + \delta\sigma_I^{(1)} + \delta\sigma_R^{(1)} \quad (43)$$

Here  $\sigma^{(0)}$  is the non-interacting contribution defined by the first term in (A6),  $\delta\sigma_I^{(1)}$  corresponds to the terms in (A6) which contain the product  $\hat{V}^+ \hat{V}^+$  ( $S_I$ -terms), while  $\delta\sigma_R^{(1)}$  is given by the terms containing  $\hat{V}^+ (1 - 2\hat{\rho}_0) \hat{V}^-$  and  $(1 - 2\hat{\rho}_0) \hat{V}^- \hat{V}^+$  ( $S_R$ -terms). As we have already shown in Ref. [5], eq. (A6) is exactly equivalent to one derived by AAG [4]. In particular, it contains the combination  $\coth \frac{\hbar\omega}{2T} + \tanh \frac{\epsilon - \hbar\omega}{2T}$  leading to partial cancellation of the terms  $\delta\sigma_I^{(1)}$  and  $\delta\sigma_R^{(1)}$ .

The perturbative result (A6) is reproduced at every stage of our analysis.

- In order to obtain (A6) one can just evaluate the operators  $\hat{U}_{1,2}$  perturbatively starting directly from their definition (16). Substituting the result in eq. (14) and replacing  $\hat{\rho}_V(t') \rightarrow \hat{\rho}_0$  in (14) one arrives at (A6).
- Alternatively, one can also expand the exact expressions for  $\hat{U}_{1,2}$  (24) in  $V^\pm$ . One recovers the same result (A6).
- One can also expand approximate expressions for  $\hat{U}_{1,2}$  (39) with the same result.
- Finally, one can perform a perturbative expansion of the quasiclassical result (40). One should expand  $J^{nm}$  to the first order in  $S_I$  and  $A_{1,2}^{nm}$  to the first order in  $V_0$ . One obtains terms proportional to the functions  $\langle V^+ V^+ \rangle \rightarrow I$  (A3) and  $\langle V^+ V^- \rangle \rightarrow R$  (A4). The structure of this first order quasiclassical result is identical to that of eq. (A6), in the latter one should just use the quasiclassical form (29) for the matrix elements of the

operators  $\hat{u}_{1,2}$  and replace the coordinates in the arguments of the fields  $V^+$  by the classical paths,  $V^+(\tau_j, x_j) \rightarrow V^+(\tau_j, \tilde{x}_{1n,2m}(\tau_j))$ . Note that this substitution should be performed neither for the field  $V^-$  nor for the electron density matrix  $\hat{\rho}_0$  because no quasiclassical approximation was employed with this matrix. Further details are presented in Appendix.

### E. Pauli principle and dephasing in the ground state

It is sometimes argued that electron decoherence at  $T = 0$  is impossible because of the Pauli principle: Electron at the Fermi surface can neither lose nor gain energy, hence, it cannot decohere. Cancellation of “coth” and “tanh” terms in the first order perturbation theory is considered by some authors as a formal consequence of this energy constraint and, on the contrary, independence of  $\tau_\varphi$  on “tanh” terms is interpreted as a sign of physical inconsistency of the calculation (“Pauli principle is lost by approximations”).

Our analysis – which fully accounts for the Pauli principle – does not support the above point of view. Our final result, eqs. (40)-(42), does depend on the Fermi function, however, this dependence enters only into the pre-exponent via the matrix elements  $A_{1,2}^{nm}$  (42) which in turn depend on the electron density matrix  $\hat{\rho}_0$ . Thus, the Pauli principle does not have any significant impact on the dephasing process. As we have already explained in Ref. [3] and elsewhere, electron dephasing at low  $T$  is only caused by fluctuations of the field  $V^+$ . Such fluctuations are described by the  $S_I$ -term in the effective action which is not sensitive to  $\rho_V$  at all. In the presence of interactions the electron energy fluctuates and it remains conserved only on average. At the same time electrons cannot, of course, infinitely decrease their energies. Within our formalism such process is prevented by the dissipative terms which explicitly depend on  $\rho_V$ . For instance, eq. (99) of Ref. [3] demonstrates that electrons above the Fermi level decrease energies, however, for energies below  $\mu$  effective “damping” produced by the electron bath becomes negative, and the holes are pushed up to the Fermi surface. Such processes give rise to the time dependence of the pre-exponent contained in the matrix elements  $A_{1,2}^{nm}$  (42).

In the arguments against quantum dephasing at  $T = 0$  the Pauli principle is used merely as an energy constraint. Therefore such arguments are not specific to Fermi systems [22] and can be tested for any quantum particle interacting with a dissipative quantum environment. It is only important to ensure that the whole interacting system “particle+environment” is in its true ground state at  $T = 0$ . One possible way to conduct such a test is to study the equilibrium effect of persistent currents (PC) for a particle on a ring in the presence of interactions. Since nonvanishing PC can only exist in the presence of

quantum coherence, (partial) suppression of its amplitude by interactions may signal quantum dephasing.

Such a problem has recently been investigated by various authors and suppression of PC by (long range) interactions was demonstrated even at  $T = 0$  [17, 18, 24] (see also [11]). In particular, for the model of a diffusive electron gas [17, 18] one finds that PC gets suppressed by interactions exactly in the ground state provided the ring perimeter exceeds a finite dephasing length  $L_\varphi$ . This length turns out to be fully consistent with one found from our WL analysis.

Without going into further details let us briefly address only one point directly related to our discussion. A non-perturbative instanton analysis of the problem [18] demonstrates that suppression of PC by interactions is controlled by the parameter

$$\lambda \sum_{k=1}^r k a_k \sim \lambda r. \quad (44)$$

Here  $\lambda = 3/(8k_F^2 l^2) \ll 1$  is the dimensionless interaction strength,  $a_k$  are the Fourier coefficients of the interaction kernel ( $a_k \sim (2/\pi r) \ln(r/k)$  for  $1 \leq k \lesssim r$  and  $a_k \approx 0$  otherwise) and  $r = R/l \gg 1$  with  $R$  being the ring radius. Provided the parameter (44) is large, PC is strongly suppressed even at  $T = 0$ . The dephasing length  $L_\varphi$  is derived from the condition  $\lambda r \sim 1$  which yields [18]  $L_\varphi \sim l/\lambda$ .

Up to a numerical prefactor the parameter (44) is just the instanton action describing tunneling between two different topological sectors of the problem. The result (44) cannot be correctly reproduced within the perturbation theory. Indeed, let us expand the flux-depending part of the free energy and PC to the first order in  $\lambda$ . Then at  $T = 0$  PC is found to be proportional to the following combination [18]

$$\phi_x - \frac{\lambda}{2} \sum_{k=1}^r k a_k \ln \left( \frac{k + 2\phi_x}{k - 2\phi_x} \right), \quad (45)$$

where  $-0.5 < \phi_x \leq 0.5$  is the external flux (normalized to the flux quantum) piercing the ring. For small  $\phi_x \ll 1$  one can expand the logarithm and reduce (45) to

$$\phi_x \left[ 1 - 2\lambda \sum_{k=1}^r \frac{k a_k}{k} \right]. \quad (46)$$

The factors  $k$  in the numerator and denominator cancel and one is left only with a small correction  $2\lambda \sum_{k=1}^r a_k \sim \lambda \ll 1$ . The above cancellation in eq. (46) obtained within the Matsubara technique is to much extent analogous to “coth-tanh” cancellation in the real time approach. In both cases this cancellation is not complete, but the remaining term is small and does not give the correct answer which can only be obtained by non-perturbative means.

The above example provides yet one more illustration of insufficiency of AAV’s perturbative approach. For instance, following AAV’s logics one could qualify (44) as

“an incorrect perturbative rather than a nonperturbative” result only because this parameter is proportional to the first power of  $\lambda$  but does not agree with one derived from the perturbation theory (45). Proceeding further along these lines, one could also “highlight” a “mistake” in our non-perturbative analysis [18]. In order to do so, one could observe that the same combination  $\lambda \sum_{k=1}^r ka_k$  enters into both non-perturbative (44) and perturbative (45) expressions, however the latter also contains the logarithm which is missing in the former. Following the logics of Ref. [7] one would then be led to conclude that the logarithm “is omitted in all orders of perturbation theory” and the result (44) “is equivalent to calculating only a single contribution”  $\lambda \sum_{k=1}^r ka_k$ . In this way AAV arrived at their conclusion about missing diagrams  $\Sigma^{(c-f)}$  in our calculation.

Fortunately, a detailed Monte Carlo simulation provides a complete numerical solution for the problem [18]. It unambiguously rules out the perturbative result (45) and demonstrates that PC is indeed strongly suppressed for  $\lambda r \gg 1$  even exactly at  $T = 0$ , see figs. 1 and 2 of Ref. [18]. Similarly, our present results, eqs. (40)-(42), allow to discard perturbative calculations of the WL correction to conductivity at low temperatures.

## VI. REMARKS ON AAV’S CRITIQUE

The analysis of the previous section not only rules out the AAV’s claim about vanishing dephasing rate at  $T \rightarrow 0$  but also demonstrates that their critique of our calculation [3, 5] is irrelevant. Nevertheless, for the sake of completeness we will reply to both critical points (i) and (ii) of Ref. [7].

### A. Density matrix

In Ref. [7] AAV stated that in eqs. (43) of our paper [3] we “replace the density matrix by its “Wigner transform””, eq. (5) of Ref. [7]. This AAV’s statement is not correct. The only replacement performed in eqs. (43) of [3] as compared to the exact eqs. (40) of that paper (or eqs. (17) of this paper) is defined by our present eq. (33), where

$$\hat{\rho}_0(\hat{\mathbf{p}}, \hat{\mathbf{r}}) = n(\hat{H}_0(\hat{\mathbf{p}}, \hat{\mathbf{r}})) \quad (47)$$

and  $n(\xi) = 1/[\exp(\xi/T) + 1]$  is the Fermi function. In other words, in Ref. [3] we used the following expressions

$$1 - 2\hat{\rho}_0(\hat{\mathbf{p}}, \hat{\mathbf{r}}) = \tanh\left(\frac{\hat{\mathbf{p}}^2/2m - \mu + U_{\text{imp}}(\hat{\mathbf{r}})}{2T}\right) \quad (48)$$

and

$$\langle \mathbf{r}_1 | 1 - 2\hat{\rho}_0 | \mathbf{r}_2 \rangle = \sum_{\nu} \tanh \frac{\xi_{\nu}}{2T} \psi_{\nu}(\mathbf{r}_1) \psi_{\nu}^*(\mathbf{r}_2), \quad (49)$$

where  $\xi_{\nu}$  and  $\psi_{\nu}(\mathbf{r})$  are the eigenvalues and the eigenfunctions of the Hamiltonian  $\hat{H}_0$ . Eq. (48) was used in eq. (43) of [3] and further while constructing the effective action. Eq. (49) was used in Section 4 and Appendix A of Ref. [5] (cf. eq. (54) of that paper) while performing the first order perturbative calculation of the conductance. With the aid of the form (49) in Ref. [5] we have proven (partial) cancellation of “coth” and “tanh” terms in the first order at  $T \rightarrow 0$  and reproduced the results [4]. Also the combination  $1 - 2n(\mathbf{p}, \mathbf{r})$  in eqs. (52), (54) and (68) of [3] has nothing to do with the “Wigner transform” of the density matrix, but is simply equal to  $\tanh(H_0(\mathbf{p}, \mathbf{r})/2T)$ . This form yields purely real  $S_R$  for all paths and  $S_R = 0$  for any pair of time-reversed classical paths.

Let us compare our eqs. (47)-(49) with eq. (5) of Ref. [7]. The latter equation [7] defines an object,  $\rho_{1'4}$ , which is neither an operator (cf. our eqs. (47,48)) nor the electron density matrix  $\rho_0(\mathbf{r}, \mathbf{r}') = \langle \mathbf{r}_1 | \hat{\rho}_0 | \mathbf{r}_2 \rangle$  (cf. our eq. (49)). We conclude that eq. (5) of Ref. [7] has nothing to do with our analysis. Since AAV’s claim of our “major mistake” and their subsequent critique are based on their eq. (5), both this claim and critique can be proven irrelevant already by a direct comparison with what was *actually* done in our paper [3].

### B. Effective action and commutation relations

In Ref. [7] AAV pointed out that while constructing our effective action we disregarded the Poisson brackets or, which is the same, the commutation relations between the operators  $\hat{\rho}_0$  and  $\hat{V}^-$  entering the Hamiltonians (17). AAV furthermore argued that if one takes care about ordering of these operators, one arrives at the effective action different from ours. Although the form of this action was not specified, it was claimed in Ref. [7] that the term  $S_R$  is not anymore real, but contains an imaginary part. According to AAV this imaginary part provides nonzero contribution to  $S_R$  evaluated on pairs of time reversed paths and “in perturbation theory ensures that the ultraviolet divergence in  $iS_R$  cancels that of  $S_I$ ”.

The latter statement of AAV is false. The correct one is just the opposite: It is the *real* part of  $S_R$  that gives rise to “tanh”-terms which compensate “coth”-contributions in the first order perturbation theory at  $T = 0$ . AAV seem not to appreciate the fact that the term  $iS_R$  in the exponent of the influence functional and the matrix elements generated by this term in the perturbation theory are different mathematical objects. The perturbative contribution from a purely *imaginary* term  $iS_R$  can and does cancel the contribution from a purely *real* term  $S_I$  in the first order at  $T \rightarrow 0$  within the Golden rule approximation. This is a general property not specific to any particular calculation. For more information we refer to the textbook [15] where the derivation of the perturbation theory from the influence functional was analyzed in details, see eqs. (12-104) to (12-108) of that book.

As to the commutation relations, everything is, of

course, correct with them in our path integral analysis. In order to demonstrate that one should only keep track of correct ordering for the operators in the perturbation expansion. One way could be to proceed directly with the Hamiltonians (17) where ordering is defined uniquely and no ambiguity can occur. Alternatively, the full perturbation theory can be recovered by expanding the influence functional in powers of  $iS_R + S_I$ . In this case one should (a) replace the momentum and coordinate variables by the operators  $p \rightarrow \hat{p}$ ,  $r \rightarrow \hat{r}$  and (b) specify the proper way of ordering (fixed by eqs. (17)) *in addition* to the expression for the effective action [25].

Furthermore, as it was demonstrated above, in order to find  $\tau_\varphi$  at  $T \rightarrow 0$  it is sufficient to correctly derive the classical ( $\hbar$ -independent) part of the action only. Obviously, while deriving  $S^{(\text{cl})}$  there is no need to take care of the commutation relations at all. This action is always real and is obtained from the quantum Hamiltonian by replacing the operators by the corresponding  $c$ -number functions. For instance, from (17) (after the replacement  $\hat{\rho}_V(t) \rightarrow \hat{\rho}_0$ ) we obtain

$$S_{1,2}^{(\text{cl})} = \int_{t'}^t d\tau \left[ p\dot{x} - \frac{p^2}{2m} - U_{\text{imp}}(x) + eV^+(\tau, x) \right. \\ \left. \pm \frac{1}{2}[1 - 2n(H_0(p, x))]eV^-(\tau, x) \right]. \quad (50)$$

These actions are real and insensitive to the ordering of  $e\hat{V}^-$  and  $1 - 2\hat{\rho}_0$ . Hence, the action  $S_R$  [3] obtained from  $S_{1,2}^{(\text{cl})}$  by averaging over  $V^\pm$ , is real as well. On top of that,  $S_R$  vanishes on pairs of time-reversed paths. Hence, it can only contribute to the pre-exponent. The latter represents the quantum correction which is sensitive to the ordering of the operators. However, this correction is formally smaller in the parameter  $\hbar$ , and, as we have already discussed in Sec. 4, it can never cancel  $S^{(\text{cl})}$  as long as the latter exceeds  $\hbar$ .

In our problem it is not convenient to apply the van Vleck formula (29) directly to the Hamiltonians  $H_{1,2}$  (17). This is because the latter contain the sharp function of the electron momentum  $1 - 2n(H_0(p, x))$  which effectively turns fluctuations around the classical paths non-Gaussian. This – purely technical – complication is circumvented by eqs. (24) and the subsequent analysis of Sec. 5. This analysis demonstrates that non-Gaussian fluctuations give rise to the pre-exponential factors  $A_{1,2}$  in the expression for the conductivity (40). We have proven in Sec. 5 that these factors are irrelevant for dephasing because they do not grow at long times and, hence, cannot cancel the term  $S_I$ .

## VII. SUMMARY

In summary, we have derived a complete expression for the weak localization correction to the conductivity of a disordered conductor in the presence electron-electron interactions. Our analysis has been carried out within

the standard model for an interacting electron gas in disordered conductors with no approximations beyond the accuracy of the definition of the WL correction. In particular, interactions have been treated non-perturbatively, no quasiclassical approximation for the electron density matrix has been employed and no disorder averaging has been performed at all. We have fully confirmed our earlier results [2, 3, 5, 6] and extended them by explicitly taking into account quantum fluctuations around the classical paths for interacting electrons. We have proven that such fluctuations, while practically irrelevant for the calculation of  $\tau_\varphi$ , do contribute to the Cooperon dynamics at short times causing, for instance, partial cancellation of the well known “coth” and “tanh” terms in the first order perturbation theory. We have also demonstrated the failure of a perturbative calculation [7] in the problem of quantum dephasing of electrons at low temperatures. Finally we have refuted AAV’s critique of our previous calculation [3, 5] observing that (i) Poisson brackets are irrelevant for the problem of electron dephasing by interactions and (ii) no “Wigner transform” of the electron density matrix was performed in our derivation.

## VIII. ACKNOWLEDGMENT

We are grateful to G. Schön for numerous instructive discussions. This work is part of the Kompetenznetz “Funktionelle Nanostrukturen” supported by the Landestiftung Baden-Württemberg gGmbH.

*Note added.* After this paper had already been submitted there appeared an independent work [26] addressing the same issue. In this work von Delft (vD) has successfully re-derived our influence functional for interacting electrons [3] and argued that (a) our approach “properly incorporates the Pauli principle” and (b) “the standard Keldysh diagrammatic expressions for the self energy of the Cooperon can be obtained from  $iS_R + S_I$ ”, i.e. from our influence functional. Thus, it is now verified not only in our work but also independently by other authors [26, 27] that our path integral result (10) [3] contains *all* RPA diagrams to *all* orders in the electron-electron interaction.

The observations (a) and (b) are important because they allow to restrict the whole discussion to just one – purely mathematical – issue, i.e. how to correctly evaluate the path integral (10). We believe that the analysis presented in Sec. 5 of this paper should eliminate all doubts [26] concerning the role of  $S_R$ -terms for quantum dephasing of electrons at low temperatures. This analysis, for instance, involves *none* of the approximations denoted in [26] as (i), (ii) and (iii). In particular, it rules out vD’s conjecture that within our approach we “neglect all the diagrams of Fig. 2b” of [26]. Quite on the contrary, our final result, eqs. (40)-(42), explicitly accounts for all these diagrams (giving rise to the “tanh”-contribution (A9) in the first order) as well as for infinitely many diagrams of all higher orders not presented in Ref. [26].

Several additional comments are in order: (i) The statement [26] that the first order perturbative result contains no ultraviolet (UV) divergences is explicitly incorrect for 1d and 2d systems, see, e.g., eq. (70) of Ref. [5]. (ii) We disagree with vD's conjecture (see the footnote 16 in Ref. [26]) that diagrams with crossed and overlapping interaction lines can be neglected [28]. (iii) Further evidence that the above conjecture is problematic is provided by the results [18] which we also address in Sec. 5E [29]. In that problem the first order diagrams yield negligible contribution (46) and, hence, the correct result (44) is dominated by *all* the remaining diagrams with crossed and overlapping interaction lines. (iv) The argument presented by eqs. (10-11) of [26] is inconclusive, since it is based on an improper application of quasiclassical methods to Fermi systems [30]. If one expands the electron density matrix  $\rho_0$  in powers of  $\hbar$  one indeed gets a series of terms diverging at  $T \rightarrow 0$ . However, this observation can only imply that the Taylor expansion of the step function (i.e. the Fermi function at  $T \rightarrow 0$  and energies close to  $\mu$ ) is essentially useless. A much more useful strategy is to retain  $\rho_0$  (which is, of course, always finite and not large) in its full quantum mechanical form and to apply quasiclassics only to those matrix elements which do not contain  $\rho_0$ . This strategy was implemented in Sec. 5. (v) In contrast to vD's conjecture in the footnote 23 of [26] our result (40)-(42) does *not* diverge at  $T \rightarrow 0$ .

## APPENDIX A

Here we will summarize several expressions used in our calculation and present some perturbative in the interaction results for the conductivity.

Averaging over the fluctuating fields  $V^\pm$  implies calculating the double path integral

$$\langle \dots \rangle_V = \int \mathcal{D}V^+ \mathcal{D}V^- (\dots) \exp \left\{ \frac{i}{\hbar} S_{EM}[V^\pm] \right\}. \quad (\text{A1})$$

Here  $S_{EM}$  can be understood as a formally exact effective action for the Hubbard-Stratonovich fields  $V^\pm$ , see, e.g., eq. (10) of Ref. [3]. For the situation discussed here it is sufficient to expand  $S_{EM}$  to the second order in  $V$ . Then one finds

$$S_{EM}[V^\pm] = \int \frac{d^4 K}{(2\pi)^4} V^-(K) \frac{k^2 \epsilon(K)}{4\pi} V^+(K) + \frac{i}{2} \int \frac{d^4 K}{(2\pi)^4} V^-(K) \frac{k^2 \text{Im}\epsilon(K)}{4\pi} \coth \frac{\hbar\omega}{2T} V^-(K), \quad (\text{A2})$$

where  $K = (\omega, k)$ . This action allows to determine the correlation functions

$$\begin{aligned} \langle V^+(t, r) V^+(0, 0) \rangle &= \hbar I(t, r) \\ &= \hbar \int \frac{d^4 K}{(2\pi)^4} \text{Im} \left( \frac{-4\pi}{k^2 \epsilon(K)} \right) \coth \frac{\hbar\omega}{2T} e^{-iKX}, \quad (\text{A3}) \end{aligned}$$

$$\begin{aligned} \langle V^+(t, r) V^-(0, 0) \rangle &= i\hbar R(t, r) \\ &= i\hbar \int \frac{d^4 K}{(2\pi)^4} \frac{4\pi}{k^2 \epsilon(K)} e^{-iKX}, \quad (\text{A4}) \\ \langle V^-(t, r) V^-(0, 0) \rangle &= 0. \quad (\text{A5}) \end{aligned}$$

Here we have defined  $X = (t, r)$ , and  $KX = \omega t - kr$ . Employing the action (A2) is equivalent to describing the electron-electron interaction within the random phase approximation.

Let us expand the results (14,24) to the first non-vanishing order in the interaction (second order in  $V^\pm$ ) and replace  $\hat{\rho}_V(t') = \hat{\rho}_0$ . Then we obtain

$$\begin{aligned} \sigma &= -\frac{ie}{3\hbar} \int_{-\infty}^t dt' \left\langle \text{tr} \left( \hat{j}(x) \hat{u}_0(t, t') [\hat{x}, \hat{\rho}_0] \hat{u}_0(t', t) \right) \right\rangle_V \\ &- \frac{2e^3}{3\hbar^3} \int_{-\infty}^t dt' \int_{t'}^t d\tau_1 \int_{t'}^{\tau_1} d\tau_2 \text{Im} \left[ \left\langle \text{tr} \left( \hat{j}(x) \hat{u}_0(t, \tau_1) \right. \right. \right. \\ &\times \hat{V}^+(\tau_1) \hat{u}_0(\tau_1, \tau_2) \left\{ \hat{V}^+(\tau_2) + \frac{1}{2}(1 - 2\hat{\rho}_0) \hat{V}^-(\tau_2) \right\} \\ &\times \hat{u}_0(\tau_2, t') [\hat{x}, \hat{\rho}_0] \hat{u}_0(t', t) \left. \right. \left. \right) \\ &+ \text{tr} \left( \hat{j}(x) \hat{u}_0(t, \tau_2) \left\{ \hat{V}^+(\tau_2) + \frac{1}{2}(1 - 2\hat{\rho}_0) \hat{V}^-(\tau_2) \right\} \right. \\ &\left. \left. \hat{u}_0(\tau_2, t') [\hat{x}, \hat{\rho}_0] \hat{u}_0(t', \tau_1) \hat{V}^+(\tau_1) \hat{u}_0(\tau_1, t) \right) \right] \right\rangle_V \quad (\text{A6}) \end{aligned}$$

After averaging over  $V^\pm$  the latter expression coincides with the result derived in Ref. [4]. The term proportional to  $\coth \frac{\hbar\omega}{2T}$  emerges from the average  $\langle \hat{V}^+(\tau_1) \hat{V}^+(\tau_2) \rangle$ , while  $\tanh \frac{\hbar\omega}{2T}$  appears from the combination  $1 - 2\hat{\rho}_0$ . Further details can be found in Ref. [5].

Note, that eq. (A6) represents the exact first order result obtained without any evaluation of the path integrals. Eq. (40) is valid to all orders in the interaction, but it was derived by evaluating the path integrals in the quasiclassical limit  $k_F l \gg 1$ . Let us expand  $J^{nm}$  (41) to the first order in  $S_I$  and  $A_{1,2}^{nm}$  (42) to the first order in  $V_0$ . Then with the aid of eq. (40) we reproduce eq. (A6) with trivial modifications as described towards the end of Sec. 5D.

Let us now identically transform our first order quasiclassical results for  $\delta\sigma_{I,R}^{(1)}$  to a somewhat different form. For that purpose let us express the electron density matrix  $\rho_0$  as follows:

$$\begin{aligned} \langle z_1 | \hat{\rho}_0 | z_2 \rangle &= \rho_0(z_1, z_2) \\ &= \frac{1}{2} \int_{-\infty}^{+\infty} ds_2 \left[ \delta(s_2) + \frac{iT}{\hbar \sinh[\frac{\pi T s_2}{\hbar}]} \right] u_0(-s_2, z_1, z_2), \quad (\text{A7}) \end{aligned}$$

where  $u_0(-s, z_1, z_2) = \langle z_1 | \hat{u}_0(0, s) | z_2 \rangle$  is the matrix element of the evolution operator for non-interacting electrons. In addition we note that the functions  $I(t, r)$  (A3)

and  $R(t, r)$  (A4) are related to each other by means of the identity

$$I(X) = \int ds_1 \frac{T \coth \frac{\pi T s_1}{\hbar}}{2\hbar} [R(t - s_1, r) + R(-t - s_1, r)].$$

Making use of the above identities we arrive at the following expressions for  $\delta\sigma_{\text{I,R}}^{(1)}$ :

$$\begin{aligned} \delta\sigma_1^{(1)} = & \frac{e^4}{12m\hbar} \sum_{n,m} \int_{-\infty}^t dt' \int_{-\infty}^{+\infty} ds_1 ds_2 \int dy_1 dy_2 \frac{-iT^2 \coth[\pi T s_1/\hbar]}{\hbar^2 \sinh[\pi T s_2/\hbar]} (\nabla_{x_1} - \nabla_{x_2})|_{x_1=x_2} \\ & \times \frac{1}{(2\pi\hbar)^3} \sqrt{\det \frac{\partial^2 S_0^{(n)}(t, t', x_1, y_1)}{\partial x_1 \partial y_1} \det \frac{\partial^2 S_0^{(m)}(t, t', x_2, y_2)}{\partial x_2 \partial y_2}} \\ & \times \exp \left\{ \frac{i}{\hbar} S_0^{(n)}(t, t', x_1, y_1) - \frac{i}{\hbar} S_0^{(m)}(t, t', x_2, y_2) \right\} \\ & \times \left\{ \int_{t'}^t d\tau_1 \int_{t'}^t d\tau_2 \left[ R(\tau_1 - \tau_2 - s_1, \tilde{x}_{1n}(\tau_1) - \tilde{x}_{1n}(\tau_2)) + R(\tau_1 - \tau_2 - s_1, \tilde{x}_{2m}(\tau_1) - \tilde{x}_{2m}(\tau_2)) \right. \right. \\ & \left. \left. - R(\tau_1 - \tau_2 - s_1, \tilde{x}_{1n}(\tau_1) - \tilde{x}_{2m}(\tau_2)) - R(\tau_1 - \tau_2 - s_1, \tilde{x}_{2m}(\tau_1) - \tilde{x}_{1n}(\tau_2)) \right] \right\} \\ & \times (y_1 - y_2) u_0(-s_2, y_1, y_2) \end{aligned} \quad (\text{A8})$$

and

$$\begin{aligned} \delta\sigma_{\text{R}}^{(1)} = & \frac{e^4}{12m\hbar} \sum_{n,m} \int_{-\infty}^t dt' \int_{-\infty}^{\infty} ds_1 ds_2 \int dy_1 dy_2 dz dr \frac{T}{\hbar \sinh[\pi T s_1/\hbar]} \frac{-iT}{\hbar \sinh[\pi T s_2/\hbar]} \\ & \times (\nabla_{x_1} - \nabla_{x_2})|_{x_1=x_2} \frac{1}{(2\pi\hbar)^3} \sqrt{\det \frac{\partial^2 S_0^{(n)}(t, t', x_1, y_1)}{\partial x_1 \partial y_1} \det \frac{\partial^2 S_0^{(m)}(t, t', x_2, y_2)}{\partial x_2 \partial y_2}} \\ & \times \exp \left\{ \frac{i}{\hbar} S_0^{(n)}(t, t', x_1, y_1) - \frac{i}{\hbar} S_0^{(m)}(t, t', x_2, y_2) \right\} \\ & \times \left\{ \int_{t'}^t d\tau_1 \int_{t'}^t d\tau_2 \left[ u_0(t' - \tau_2 - s_1, y_1, r) [R(\tau_1 - \tau_2, \tilde{x}_{1n}(\tau_1) - r) \right. \right. \\ & \left. \left. - R(\tau_1 - \tau_2, \tilde{x}_{2m}(\tau_1) - r)] u_0(\tau_2 - t', r, z) (z - y_2) u_0(-s_2, z, y_2) \right. \right. \\ & \left. \left. - (y_1 - z) u_0(-s_2, y_1, z) u_0(t' - \tau_2, z, r) [R(\tau_1 - \tau_2, \tilde{x}_{1n}(\tau_1) - r) \right. \right. \\ & \left. \left. - R(\tau_1 - \tau_2, \tilde{x}_{2m}(\tau_1) - r)] u_0(\tau_2 - t' - s_1, r, y_2) \right] \right\}. \end{aligned} \quad (\text{A9})$$

Eqs. (A8) and (A9) have a very similar structure. These two expressions are, however, not fully identical even at  $T \rightarrow 0$  and, hence, they *do not* cancel exactly in the first order result (43), see also Sec. 4 of Ref. [5].

Further evaluation of eqs. (A8) and (A9) makes little sense because the first order perturbation theory cannot provide any useful information about the electron dephasing time at low temperatures. Nevertheless, the above expressions are of a certain interest, since they help to illustrate the relation between perturbative and non-perturbative results at the stage when the quasiclassical approximation has already been performed. We observe, for instance, that all the first order terms, both “coth” and “tanh” contributions, are fully reproduced from our path integral analysis. Another observation concerns the

relation between the quasiclassical paths emerging from the path integrals and those entering the first order results for  $\delta\sigma_{\text{R}}^{(1)}$ . The WL correction to the conductivity is defined on pairs of time-reversed path, and only such paths (plus fluctuations around them) are relevant for the path integral analysis of this quantity. Of course, the same paths enter if the general result is expanded to the first order in the interaction before the transformation (A7). However, *after* this transformation there appear additional matrix elements  $u_0$  of the electron evolution operator. Proceeding quasiclassically, one can evaluate these matrix elements by means of the van Vleck formula (29), i.e. to write

$$u_0(t' - \tau_2 - s_1, y_1, r) \propto e^{\frac{i}{\hbar} S_0^{(k)}(t' - \tau_2 - s_1, 0, y_1, r)} \quad (\text{A10})$$

and similarly for other matrix elements. Substituting  $u_0$  in the form (A10) into eq. (A9) one can interpret the result in terms of the electron motion along additional classical paths  $\tilde{x}_k(s)$ , say, first from  $y_1$  to  $r$  and then from  $r$  to  $z$  (some of these paths violate the requirement of causality, see Fig. 3 of Ref. [5] and related discussion).

This could in turn create an illusion that these additional paths are missing in the path integral formulation. The above analysis clearly indicates the origin of such an illusion. It also demonstrates that – in contrast to Ref. [4] – the whole issue has nothing to do with disorder averaging which is not performed here at all.

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- [20] For the sake of definiteness and simplicity we use the substitution (33), in which case averaging over the fields  $V^\pm$  is performed exactly. One can also avoid this step and – as it was done in Sec. 5B – prove that averaging of the terms containing  $\hat{\rho}_V(t')$  can be performed separately from that of the matrix elements of  $\hat{u}(t, t', V^\pm)$ . For  $k_F l \gg 1$  the accuracy of this procedure is the same as that of eqs. (40)-(42).
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calculated (clearly,  $1/g$  can become large, e.g., for sufficiently long wires). Since the parameter  $1/g$  does not even exist before disorder averaging and since the diagrammatic expansion is carried out in powers of the interaction, it would be more appropriate to say that the diagrams in question contain an extra power of the effective interaction parameter, but not  $1/g$ . More importantly, an extra power of a small parameter in front of the diagram does not yet imply that its contribution is small.

[29] We disagree with vD who argued that any comparison between the models with and without the Pauli exclu-

sion principle would be “meaningless” [26]. Provided the interaction term in the Hamiltonian is the same, the two models are very similar in many respects, both physically and formally. For instance, the structure of all the diagrams is the same in all orders, and the zero temperature cancellation of (some) UV-divergent first order terms occurs in the same manner both with and without the Pauli principle (cf., e.g., eqs. (67) and (E19) of Ref. [5]).

[30] Should this argument be correct, it would equally apply to our theory and to that of AAK [21].